ISSN 2308-5878. Mathematical and computer modelling.
Series: Physical and mathematical sciences. 2026. Issue 30. P. 72-90.

UDC 512.552
DOI: 10.32626/2308-5878.2026-30.72-90

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ULTRA EXPONENT MATRICES

The paper studies ultra exponent matrices, that is, reduced exponent matrices for which the usual triangle inequality is strengthened to an ultrametric-type inequality. It is proved that every ultra exponent matrix is an exponent matrix, and that every exponent matrix whose entries belong to {0, 1} is an ultra exponent matrix. The behaviour of ultra matrices under elementary transformations is analysed: transformations of the first type do not preserve the ultra property in general, while simultaneous permutations of rows and columns do preserve it. It is shown that a quiver obtained from a reduced ultra exponent matrix with at least one entry greater than one is not rigid. A counterexample demonstrates that not every admissible quiver with a loop at every vertex can be represented by an ultra exponent matrix. Several structural characterizations are also established, including monotone deformations preserving the associated quiver, a filtration description in terms of transitive threshold relations, a connection between 0-1 ultra matrices and partial orders, a minimax interpretation, and closure under componentwise maximum. From the viewpoint of mathematical modelling, ultra exponent matrices may be interpreted as discrete directed distance models with bottleneck-type constraints, where the value assigned to a transition is determined by the strongest restriction along admissible paths rather than by an additive cost. This makes them useful for modelling hierarchical systems, priority relations, minimax optimization, constrained network flows, clustering-like structures, and algebraic or combinatorial systems whose essential information is encoded by admissible weighted quivers. The obtained results provide tools for comparing such models and for reducing redundant representations without losing the underlying directed structure.

Key words: *exponent matrix, reduced exponent matrix, admissible quiver, ultra exponent matrix, rigid quiver, weight function, mathematical modelling.*

Стаття надійшла до редакції: 14.05.2026
Рекомендовано до друку: 24.05.2026
Оприлюднено (online): 29.05.2026

Introduction. One important aspect of ring theory is the study of properties of rings by means of graph-theoretic methods. Every tiled order over a discrete valuation ring is completely determined by its exponent matrix and by the discrete valuation ring itself [1]. Many structural properties of such orders are determined by their exponent matrices, in particular by the corresponding quivers [1-3].

Exponent matrices have relatively recently become an independent object of study. In [4-8], admissible quivers, rigid quivers, unit cycles, unit quivers, and weight functions defining admissible quivers were investigated. The introduction of weight functions made it possible to formulate properties of exponent matrices in the language of weighted directed graphs.

In this paper we study ultra exponent matrices. Their main feature is that they satisfy not only the usual triangle inequality

$$\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik},$$

but also the stronger ultrametric-type inequality

$$\max\left\{\alpha_{ij}, \alpha_{jk}\right\} \geq \alpha_{ik}.$$

Throughout the paper, as is customary for exponent matrices of tiled orders, the entries of exponent matrices are assumed to be nonnegative integers.

From the viewpoint of mathematical modelling, an ultra exponent matrix can be regarded as an integer-valued directed distance table in which the distance from one vertex to another is controlled by the largest restriction on intermediate paths. Such a model is useful when a dominant transition cost, bottleneck constraint, hierarchy level, or priority relation is more significant than the sum of local costs. Therefore, ultra matrices may be applied to discrete network models, minimax optimization, clustering-type structures, and algebraic models in which admissible directed graphs encode essential structural constraints.

Basic definitions.

Definition 1.1 [1, p. 353]. *A matrix*

$$\mathcal{E} = \left(\alpha_{ij}\right) \in M_n\left(\mathbb{Z}_{\geq 0}\right)$$

is called an ***exponent matrix*** *if the following conditions hold:*

$$\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik} \text{ for all } i, j, k = 1, \ldots, n,$$

$$\alpha_{ii} = 0 \text{ for all } i = 1, \ldots, n.$$

An exponent matrix is called **reduced** if, in addition, the following condition holds:

$$\alpha_{ij} + \alpha_{ji} \geq 1 \text{ for all } i, j \in \{1, \ldots, n\},\ i \neq j.$$

Let $\mathcal{E} = \left(\alpha_{ij}\right)$ be a reduced exponent matrix. Introduce the matrix

$$\mathcal{E}^{(1)} = \left(\beta_{ij}\right) = \mathcal{E} + E_n \in M_n\left(\mathbb{Z}\right),$$

where E_n is the identity matrix. Also introduce the matrix

$$\mathcal{E}^{(2)} = \left(\gamma_{ij}\right) \in M_n\left(\mathbb{Z}\right),\ \gamma_{ij} = \min_{1 \le k \le n}\left(\beta_{ik} + \beta_{kj}\right).$$

Definition 1.2 [1, p. 357]. *The* ***quiver*** *of a reduced exponent matrix $\mathcal{E}$ is the quiver $Q = Q\left(\mathcal{E}\right)$ whose adjacency matrix is defined by the formula*

$$\left[Q\right] = \mathcal{E}^{(2)} - \mathcal{E}^{(1)}.$$

Definition 1.3 [1]. *Reduced exponent matrices $\mathcal{E}_1$ and $\mathcal{E}_2$ are called* ***equivalent*** *if one of them can be obtained from the other by a finite sequence of elementary transformations of the following two types:*

1) *subtract an integer t from all entries of the i-th row and add the same integer t to all entries of the i-th column;*
2) *interchange two rows and simultaneously interchange the two columns with the same numbers.*

Definition 1.4 [1, p. 357]. *A quiver Q is called* ***admissible*** *if there exists a reduced exponent matrix $\mathcal{E}$ such that*

$$Q\left(\mathcal{E}\right) = Q.$$

Definition 1.5. *A quiver $Q = \left(V_Q, A_Q\right)$ is called* ***weighted*** *if a function*

$$\omega : A_Q \to \mathbb{Z}$$

is defined on the set of its arrows. The function ω is called a ***weight function****, and its value on an arrow is called the weight of this arrow. The sum of the weights of all arrows along a path is called the weight of this path.*

Theorem 1 [4]. *A strongly connected quiver $Q = \left(V_Q, A_Q\right)$ is admissible if and only if there exists a weight function ω satisfying the following conditions:*

1) *the weight of an arrow from vertex i to vertex j is smaller than the weight of any path from i to j of length $l \ge 2$;*
2) *the weight of a loop at vertex i is smaller than the weight of any cycle of length $l \ge 2$ passing through vertex i;*
3) *the weight of every cycle is greater than or equal to 1;*
4) *the weight of every loop is equal to 1;*
5) *through every vertex without a loop there passes a cycle of length $l \ge 2$ whose weight is equal to 1.*

Remark 1.1. *Conditions (4) and (5) imply that through every vertex of an admissible quiver there passes a cycle of weight 1.*

Definition 1.6. *A weight function satisfying all conditions of Theorem 1 is called an* ***admissible weight function****.*

Construction. Suppose that a quiver Q and an admissible weight function ω are given. Then one can construct an exponent matrix $\mathcal{E} = \left(\alpha_{ij}\right)$ as follows: if the quiver Q contains an arrow σ_{ij}, then

$$\alpha_{ij} = \omega\left(\sigma_{ij}\right);$$

if such an arrow does not exist, then α_{ij} is defined as the weight of a lightest, that is, a minimum-weight, path from the vertex v_i to the vertex v_j.

Definition 1.7. *A simple cycle in an admissible quiver whose weight is equal to* 1 *is called a* ***unit cycle****.*

Definition 1.8 [6]. *An admissible quiver* Q *is called* ***rigid*** *if there exists, up to equivalence, a unique reduced exponent matrix* $\mathcal{E}$ *such that*

$$Q\left(\mathcal{E}\right) = Q.$$

Main results.

Definition 2.1. *A matrix* $\mathcal{E} = \left(\alpha_{ij}\right) \in M_n\left(\mathbb{Z}_{\geq 0}\right)$ *is called an* ***ultra exponent matrix*** *if the following conditions hold:*

$$\max\left\{\alpha_{ij}, \alpha_{jk}\right\} \geq \alpha_{ik} \text{ for all } i, j, k = 1, \ldots, n,$$

$$\alpha_{ii} = 0 \text{ for all } i = 1, \ldots, n.$$

If, in addition, the condition

$$\alpha_{ij} + \alpha_{ji} \geq 1 \text{ for all } i, j \in \left\{1, \ldots, n\right\},\ i \neq j$$

holds, then $\mathcal{E}$ is called a **reduced ultra exponent matrix**. In what follows, unless otherwise stated, an ultra exponent matrix means a reduced ultra exponent matrix.

Proposition 2.1. *Every ultra exponent matrix is an exponent matrix.*

Proof. Let $\mathcal{E} = \left(\alpha_{ij}\right)$ be an ultra exponent matrix. By definition,

$$\max\left\{\alpha_{ij}, \alpha_{jk}\right\} \geq \alpha_{ik}$$

for all i, j, k. Since all entries of the matrix are nonnegative, we have

$$\alpha_{ij} + \alpha_{jk} \geq \max\left\{\alpha_{ij}, \alpha_{jk}\right\}.$$

Therefore,

$$\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik}.$$

Moreover, by the definition of an ultra matrix, $\alpha_{ii} = 0$ for all i. Thus $\mathcal{E}$ satisfies all conditions in the definition of an exponent matrix. **The proposition is proved.**

Proposition 2.2. *An exponent matrix with entries* 0 *and* 1 *is an ultra exponent matrix.*

Proof. Let $\mathcal{E} = \left(\alpha_{ij}\right)$ be an exponent matrix, and suppose that all its entries belong to the set $\{0,1\}$. We must prove that, for all i, j, k, the inequality

$$\max\left\{\alpha_{ij}, \alpha_{jk}\right\} \geq \alpha_{ik}$$

holds. If $\alpha_{ik} = 0$, the inequality is obvious. Let $\alpha_{ik} = 1$. Since $\mathcal{E}$ is an exponent matrix, we have

$$\alpha_{ij} + \alpha_{jk} \geq \alpha_{ik} = 1.$$

Because $\alpha_{ij}, \alpha_{jk} \in \{0,1\}$, this inequality implies that at least one of the two entries α_{ij} and α_{jk} is equal to 1. Hence

$$\max\left\{\alpha_{ij}, \alpha_{jk}\right\} = 1 = \alpha_{ik}.$$

Thus the ultrametric inequality holds for all i, j, k. **The proposition is proved.**

Proposition 2.3. *An elementary transformation of the first type does not, in general, preserve the property of being an ultra matrix, whereas an elementary transformation of the second type preserves this property.*

Proof. First we show that a transformation of the second type preserves the property of being an ultra matrix. Such a transformation only simultaneously permutes rows and columns with the same numbers; that is, it simply renumbers the vertices. If the inequality

$$\max\left\{\alpha_{ij}, \alpha_{jk}\right\} \geq \alpha_{ik}$$

held for the original matrix, then after a simultaneous permutation of rows and columns the same inequality holds for the correspondingly renumbered indices. Hence a transformation of the second type preserves the property of being an ultra matrix.

Now we show that a transformation of the first type does not preserve this property in general. Consider the ultra matrix

$$\mathcal{E} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 2 & 0 \end{pmatrix}.$$

It is a reduced ultra exponent matrix. Perform an elementary transformation of the first type for the first row and the first column with parameter $t = -1$, that is, add 1 to all entries of the first row and subtract 1 from all entries of the first column. We obtain the matrix

$$\mathcal{E}' = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 0 \end{pmatrix}.$$

The matrix $\mathcal{E}'$ is no longer an ultra matrix. Indeed, for the indices $i=3$, $j=1$, $k=2$ we have

$$\max\{\alpha'_{31},\alpha'_{12}\}=\max\{1,1\}=1<2=\alpha'_{32}.$$

Thus the ultrametric inequality is violated. **The proposition is proved.**

Proposition 2.4. *If all off-diagonal entries of a reduced ultra exponent matrix are positive, then the quiver obtained from it is a complete quiver with a loop at every vertex.*

Proof. Let $\mathcal{E}=(\alpha_{ij})$ be a reduced ultra exponent matrix and suppose that

$$\alpha_{ij}>0 \text{ for all } i\neq j.$$

We show that, for every pair of distinct vertices i and k, the quiver $Q(\mathcal{E})$ contains an arrow $i\to k$.

For $i\neq k$, the arrow $i\to k$ is absent if and only if there exists an index $j\notin\{i,k\}$ such that

$$\alpha_{ik}=\alpha_{ij}+\alpha_{jk}.$$

However, all off-diagonal entries are positive, so

$$\alpha_{ij}+\alpha_{jk}>\max\{\alpha_{ij},\alpha_{jk}\}.$$

On the other hand, the ultra inequality gives

$$\alpha_{ik}\leq\max\{\alpha_{ij},\alpha_{jk}\}.$$

Therefore, the equality $\alpha_{ik}=\alpha_{ij}+\alpha_{jk}$ is impossible. Hence the arrow $i\to k$ exists for all $i\neq k$.

Moreover, for each i and each $j\neq i$ we have $\alpha_{ij}>0$ and $\alpha_{ji}>0$, and hence

$$\alpha_{ij}+\alpha_{ji}\geq 2.$$

This means that the quiver has a loop at every vertex. Therefore the quiver is complete and has a loop at every vertex. **The proposition is proved.**

We shall use the following auxiliary lemma in the proof of the main theorem.

Lemma 2.5. *Let $\mathcal{E}=(\alpha_{ij})$ be a reduced ultra exponent matrix. Define the function*

$$\varphi(x)=\begin{cases}0, & x=0,\\ 1, & x=1,\\ x+1, & x\geq 2,\end{cases}$$

and define the matrix $\tilde{\mathcal{E}} = \left(\tilde{\alpha}_{ij}\right)$ *by*

$$\tilde{\alpha}_{ij} = \varphi\left(\alpha_{ij}\right).$$

Then $\tilde{\mathcal{E}}$ *is a reduced ultra exponent matrix and*

$$Q\left(\tilde{\mathcal{E}}\right) = Q(\mathcal{E}).$$

Proof. Since $\varphi(0) = 0$, we have $\tilde{\alpha}_{ii} = 0$ for all i. The function φ is strictly increasing on the set of nonnegative integers. Therefore, from the inequality

$$\alpha_{ik} \le \max\left\{\alpha_{ij}, \alpha_{jk}\right\}$$

we obtain

$$\tilde{\alpha}_{ik} = \varphi\left(\alpha_{ik}\right) \le \varphi\left(\max\left\{\alpha_{ij}, \alpha_{jk}\right\}\right) = \max\left\{\varphi\left(\alpha_{ij}\right), \varphi\left(\alpha_{jk}\right)\right\} = \max\left\{\tilde{\alpha}_{ij}, \tilde{\alpha}_{jk}\right\}.$$

Hence $\tilde{\mathcal{E}}$ is an ultra matrix. Reducedness is also preserved: if $i \ne j$, then the inequality $\alpha_{ij} + \alpha_{ji} \ge 1$ implies that at least one of the entries α_{ij} and α_{ji} is positive; therefore at least one of the entries $\tilde{\alpha}_{ij}$ and $\tilde{\alpha}_{ji}$ is also positive, and hence

$$\tilde{\alpha}_{ij} + \tilde{\alpha}_{ji} \ge 1.$$

It remains to prove that the quivers $Q(\mathcal{E})$ and $Q\left(\tilde{\mathcal{E}}\right)$ coincide.

For $i \ne k$, the arrow $i \to k$ is absent in $Q(\mathcal{E})$ if and only if there exists an index $j \notin \{i, k\}$ such that

$$\alpha_{ik} = \alpha_{ij} + \alpha_{jk}.$$

We prove that this equality is preserved after applying the function φ. Suppose that

$$\alpha_{ik} = \alpha_{ij} + \alpha_{jk}.$$

Since the matrix $\mathcal{E}$ is ultra, we have

$$\alpha_{ik} \le \max\left\{\alpha_{ij}, \alpha_{jk}\right\}.$$

For nonnegative numbers, however, one always has

$$\alpha_{ij} + \alpha_{jk} \ge \max\left\{\alpha_{ij}, \alpha_{jk}\right\}.$$

Therefore the equality $\alpha_{ik} = \alpha_{ij} + \alpha_{jk}$ can hold only when one of the two summands α_{ij} and α_{jk} is zero and the other is equal to α_{ik}. Hence, after applying φ, we again have

$$\tilde{\alpha}_{ik} = \tilde{\alpha}_{ij} + \tilde{\alpha}_{jk}.$$

Conversely, if

$$\tilde{\alpha}_{ik} = \tilde{\alpha}_{ij} + \tilde{\alpha}_{jk},$$

then the ultra inequality for the matrix $\tilde{\mathcal{E}}$ implies that

$$\tilde{\alpha}_{ik} \le \max\left\{\tilde{\alpha}_{ij}, \tilde{\alpha}_{jk}\right\}.$$

Since the left-hand side is the sum of two nonnegative numbers, this is possible only when one of the summands $\tilde{\alpha}_{ij}$ and $\tilde{\alpha}_{jk}$ is zero. By injectivity of φ on $\mathbb{Z}_{\ge 0}$, we obtain the corresponding equality for the original entries:

$$\alpha_{ik} = \alpha_{ij} + \alpha_{jk}.$$

Thus, for all $i \ne k$, the absence or presence of the arrow $i \to k$ is the same in $Q(\mathcal{E})$ and in $Q\left(\tilde{\mathcal{E}}\right)$.

For loops we argue as follows: a loop at the vertex i is absent if and only if there exists $j \ne i$ such that

$$\alpha_{ij} + \alpha_{ji} = 1.$$

The function φ does not change the values 0 and 1, so the equality $\alpha_{ij} + \alpha_{ji} = 1$ holds if and only if

$$\tilde{\alpha}_{ij} + \tilde{\alpha}_{ji} = 1.$$

Therefore loops are also preserved. Consequently,

$$Q\left(\tilde{\mathcal{E}}\right) = Q(\mathcal{E}).$$

The lemma is proved.

Theorem 2. *A quiver obtained from an ultra matrix having at least one entry greater than one is not rigid.*

Proof. Let $\mathcal{E} = \left(\alpha_{ij}\right)$ be a reduced ultra exponent matrix, and suppose that there exists at least one entry $\alpha_{pq} > 1$. Construct the matrix $\tilde{\mathcal{E}}$ by the rule of Lemma 2.5:

$$\tilde{\alpha}_{ij} = \varphi\left(\alpha_{ij}\right),$$

where

$$\varphi(x) = \begin{cases} 0, & x = 0, \\ 1, & x = 1, \\ x+1, & x \ge 2. \end{cases}$$

By Lemma 2.5, the matrix $\tilde{\mathcal{E}}$ is a reduced ultra exponent matrix and gives the same quiver:

$$Q\left(\tilde{\mathcal{E}}\right) = Q(\mathcal{E}).$$

We show that the matrices $\mathcal{E}$ and $\tilde{\mathcal{E}}$ are not equivalent. Consider the sum of all entries of a matrix

$$S(\mathcal{E}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\alpha_{ij}.$$

An elementary transformation of the first type does not change this sum: if a number t is subtracted from the i-th row, the total sum decreases by nt; if the same number t is added to the i-th column, the total sum increases by nt. Thus the total change is zero. An elementary transformation of the second type merely permutes the entries of the matrix and therefore also preserves their total sum.

Hence $S(\mathcal{E})$ is an invariant of equivalence of reduced exponent matrices.

Since the matrix $\mathcal{E}$ contains an entry $\alpha_{pq} > 1$, when passing to $\tilde{\mathcal{E}}$ this entry increases by 1. All other entries either remain unchanged or also increase by 1 if they are greater than 1. Therefore

$$S(\tilde{\mathcal{E}}) > S(\mathcal{E}).$$

It follows that $\mathcal{E}$ and $\tilde{\mathcal{E}}$ cannot be equivalent. Thus the same quiver $Q(\mathcal{E})$ is obtained from at least two nonequivalent reduced exponent matrices, namely $\mathcal{E}$ and $\tilde{\mathcal{E}}$. Therefore this quiver is not rigid. **The theorem is proved.**

The following result shows that the assertion that every admissible quiver with a loop at every vertex can be obtained from an ultra matrix is false in general.

Theorem 3. *There exists an admissible quiver with a loop at every vertex that cannot be obtained from any ultra exponent matrix.*

Proof. Consider the quiver Q with the set of vertices

$$V_Q = \{1, 2, 3, 4\}$$

and the adjacency matrix

$$[Q] = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

This quiver has a loop at every vertex. Moreover, it contains the arrows

$$1 \leftrightarrow 2,\ 2 \leftrightarrow 3,\ 3 \leftrightarrow 4,$$

and contains no other non-loop arrows.

First we show that Q is admissible. Define a weight function ω by assigning weight 1 to every arrow, including the loops:

$$\omega(\sigma) = 1 \text{ for every arrow } \sigma \in A_Q.$$

Then the weight of any path of length l is equal to l. In particular, the weight of every arrow is equal to 1 and is smaller than the weight of any path of length $l \geq 2$. The weight of every loop is also equal to 1 and is smaller than the weight of any cycle of length $l \geq 2$. The weight of every cycle is greater than or equal to 1, and the weight of every loop is equal to 1. Since every vertex has a loop, the condition concerning vertices without loops is satisfied automatically. By Theorem 1, the quiver Q is admissible.

Now we prove that Q cannot be obtained from an ultra matrix. Suppose, to the contrary, that there exists a reduced ultra exponent matrix

$$\mathcal{E} = (\alpha_{ij})$$

such that

$$Q(\mathcal{E}) = Q.$$

Denote the weights of the arrows along the path $1 \to 2 \to 3 \to 4$ by

$$a = \alpha_{12},\ b = \alpha_{23},\ c = \alpha_{34},$$

and the weights of the arrows in the opposite direction by

$$a' = \alpha_{21},\ b' = \alpha_{32},\ c' = \alpha_{43}.$$

Since the quiver Q has no arrow $1 \to 4$, while the only simple directed path from vertex 1 to vertex 4 is

$$1 \to 2 \to 3 \to 4,$$

the corresponding matrix entry must be equal to the weight of the lightest path from 1 to 4:

$$\alpha_{14} = a + b + c.$$

On the other hand, since $\mathcal{E}$ is an ultra matrix, we successively obtain

$$\alpha_{13} \leq \max\{a, b\},$$

and also

$$\alpha_{14} \leq \max\{\alpha_{13}, c\} \leq \max\{a, b, c\}.$$

Hence

$$a + b + c = \alpha_{14} \leq \max\{a, b, c\}.$$

But a, b, c are nonnegative integers, so always

$$a + b + c \geq \max\{a, b, c\}.$$

Thus

$$a + b + c = \max\{a, b, c\}.$$

This is possible only if at least two of the numbers a, b, c are equal to zero.

Analogously, since Q has no arrow $4 \to 1$, while the only simple directed path from 4 to 1 is

$$4 \to 3 \to 2 \to 1,$$

we obtain

$$\alpha_{41} = c' + b' + a'.$$

The ultra inequality implies

$$\alpha_{41} \le \max\{c', b', a'\}.$$

Therefore at least two of the numbers a', b', c' are equal to zero.

Consequently, among the three pairs

$$(a, a'),\ (b, b'),\ (c, c')$$

there is at least one pair in which both entries are equal to zero. Indeed, among a, b, c there are at least two zeros, and among a', b', c' there are also at least two zeros; therefore, for some adjacent pair of vertices, we simultaneously have

$$\alpha_{ij} = 0 \text{ and } \alpha_{ji} = 0.$$

This contradicts the reducedness of the matrix, since for $i \ne j$ one must have

$$\alpha_{ij} + \alpha_{ji} \ge 1.$$

Thus the assumption that there exists an ultra matrix $\mathcal{E}$ with $Q(\mathcal{E}) = Q$ is false. Therefore there exists an admissible quiver with a loop at every vertex that cannot be obtained from an ultra exponent matrix. **The theorem is proved.**

Additional structural properties of ultra matrices. In this section we present several additional results that clarify the internal structure of ultra exponent matrices and show that they can be described not only by an ultrametric-type inequality, but also by transitive relations, minimax operations, and monotone deformations of values.

Theorem 4.1. *Let* $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z}_{\ge 0})$ *be a reduced ultra exponent matrix. Let*

$$f : \mathbb{Z}_{\ge 0} \to \mathbb{Z}_{\ge 0}$$

be a strictly increasing function such that

$$f(0) = 0,\ f(1) = 1.$$

Construct the matrix

$$f(\mathcal{E}) = \left(f(\alpha_{ij})\right).$$

Then $f(\mathcal{E})$ *is a reduced ultra exponent matrix and*

$$Q\big(f(\mathcal{E})\big) = Q(\mathcal{E}).$$

Proof. Since $\mathcal{E}$ is an ultra matrix, for all i, j, k we have

$$\alpha_{ik} \le \max\left\{\alpha_{ij}, \alpha_{jk}\right\}.$$

The function f is strictly increasing, hence

$$f(\alpha_{ik}) \le f\left(\max\left\{\alpha_{ij}, \alpha_{jk}\right\}\right).$$

Since f is increasing, we have

$$f\left(\max\left\{\alpha_{ij}, \alpha_{jk}\right\}\right) = \max\left\{f\left(\alpha_{ij}\right), f\left(\alpha_{jk}\right)\right\}.$$

Therefore,

$$f(\alpha_{ik}) \le \max\left\{f\left(\alpha_{ij}\right), f\left(\alpha_{jk}\right)\right\}.$$

This means that the matrix $f(\mathcal{E})$ satisfies the ultrametric inequality. Moreover, since $\alpha_{ii} = 0$ and $f(0) = 0$, we have

$$f(\alpha_{ii}) = 0$$

for all i. Hence $f(\mathcal{E})$ is an ultra matrix.

Let us verify reducedness. For $i \ne j$, we have

$$\alpha_{ij} + \alpha_{ji} \ge 1.$$

Since the entries are nonnegative, at least one of the entries α_{ij} and α_{ji} is not less than 1. By strict monotonicity of f and the equality $f(1) = 1$, the corresponding entry after applying f is also not less than 1. Therefore

$$f\left(\alpha_{ij}\right) + f\left(\alpha_{ji}\right) \ge 1.$$

Thus $f(\mathcal{E})$ is a reduced ultra exponent matrix.

It remains to prove that the quiver does not change. For distinct i and k, the arrow $i \to k$ is absent in $Q(\mathcal{E})$ if and only if there exists an index $j \notin \{i, k\}$ such that

$$\alpha_{ik} = \alpha_{ij} + \alpha_{jk}.$$

In the case of an ultra matrix, such an equality is possible only when one of the summands α_{ij} and α_{jk} is zero. Indeed, if both summands are positive, then

$$\alpha_{ij} + \alpha_{jk} > \max\left\{\alpha_{ij}, \alpha_{jk}\right\},$$

whereas the ultra inequality gives

$$\alpha_{ik} \le \max\left\{\alpha_{ij}, \alpha_{jk}\right\},$$

which contradicts the equality $\alpha_{ik} = \alpha_{ij} + \alpha_{jk}$. Hence, if such an equality holds, one summand is zero and the other is equal to α_{ik}. Since $f(0) = 0$, after applying f we obtain

$$f(\alpha_{ik}) = f(\alpha_{ij}) + f(\alpha_{jk}).$$

Conversely, if

$$f(\alpha_{ik}) = f(\alpha_{ij}) + f(\alpha_{jk}),$$

then, applying the already proved ultra inequality to the matrix $f(\mathcal{E})$, we again obtain that one summand on the right-hand side is zero. Because $f(0) = 0$ and f is strictly increasing, the corresponding original summand is also zero. Thus the decomposition equality for $\mathcal{E}$ and for $f(\mathcal{E})$ holds simultaneously.

Hence all non-loop arrows in the quivers $Q(\mathcal{E})$ and $Q(f(\mathcal{E}))$ coincide.

For loops the preservation is analogous. A loop at vertex i is absent if and only if there exists $j \ne i$ such that

$$\alpha_{ij} + \alpha_{ji} = 1.$$

Since $f(0) = 0$, $f(1) = 1$, and strict monotonicity implies $f(x) > 1$ for $x \ge 2$, an equality with sum 1 is preserved if and only if it held before applying f. Thus loops are also preserved. Therefore

$$Q(f(\mathcal{E})) = Q(\mathcal{E}).$$

The theorem is proved.

Remark 4.2. *Lemma 2.5 is a special case of Theorem 4.1 for the function*

$$f(x) = \begin{cases} 0, & x = 0, \\ 1, & x = 1, \\ x+1, & x \ge 2. \end{cases}$$

This special case is used in the proof of non-rigidity of quivers obtained from ultra matrices having at least one entry greater than one.

Theorem 4.3. *Let $\mathcal{E} = (\alpha_{ij}) \in M_n(\mathbb{Z}_{\ge 0})$ be a matrix with zero main diagonal. For each integer $r \ge 0$, define a relation R_r on the set $\{1, 2, \ldots, n\}$ by*

$$iR_r j \Leftrightarrow \alpha_{ij} \le r.$$

Then $\mathcal{E}$ is an ultra exponent matrix if and only if, for every $r \ge 0$, the relation R_r is transitive.

Proof. Suppose first that $\mathcal{E}$ is an ultra matrix. Assume that $iR_r j$ and $jR_r k$. This means that

$$\alpha_{ij} \le r,\ \alpha_{jk} \le r.$$

Then

$$\max\left\{\alpha_{ij}, \alpha_{jk}\right\} \le r.$$

By the ultra inequality,

$$\alpha_{ik} \le \max\left\{\alpha_{ij}, \alpha_{jk}\right\} \le r.$$

Hence $iR_r k$. Thus R_r is transitive for every $r \ge 0$.

Conversely, suppose that for every $r \ge 0$ the relation R_r is transitive. Take arbitrary indices i, j, k and put

$$r = \max\left\{\alpha_{ij}, \alpha_{jk}\right\}.$$

Then

$$\alpha_{ij} \le r,\ \alpha_{jk} \le r,$$

that is, $iR_r j$ and $jR_r k$. By transitivity of R_r, we have $iR_r k$, that is,

$$\alpha_{ik} \le r = \max\left\{\alpha_{ij}, \alpha_{jk}\right\}.$$

This is precisely the ultra inequality. Therefore $\mathcal{E}$ is an ultra matrix. The theorem is proved.

Corollary 4.4. *If $\mathcal{E} = \left(\alpha_{ij}\right)$ is a reduced ultra exponent matrix, then the relation $\le$ on the set $\left\{1, 2, \ldots, n\right\}$ defined by*

$$i \le j \Leftrightarrow \alpha_{ij} = 0$$

is a partial order.

Proof. Reflexivity follows from the equality $\alpha_{ii} = 0$. Transitivity follows from Theorem 4.3 for $r = 0$: if $\alpha_{ij} = 0$ and $\alpha_{jk} = 0$, then $\alpha_{ik} = 0$. Let us verify antisymmetry. If for $i \ne j$ the equalities

$$\alpha_{ij} = 0,\ \alpha_{ji} = 0$$

held simultaneously, then

$$\alpha_{ij} + \alpha_{ji} = 0,$$

which contradicts the reducedness of the matrix. Hence $\le$ is a partial order. **The corollary is proved.**

Corollary 4.5. *Reduced ultra exponent matrices with entries* 0 *and* 1 *are in one-to-one correspondence with partial orders on the set* $\{1,2,\ldots,n\}$.

More precisely, if a partial order $\leq$ *is given on the set* $\{1,2,\ldots,n\}$, *then the corresponding matrix is defined by*

$$\alpha_{ij}=\begin{cases}0, & i\leq j,\\ 1, & i\not\leq j.\end{cases}$$

Proof. Let $\mathcal{E}$ be a reduced $0,1$-ultra matrix. By Corollary 4.4, the relation

$$i\leq j\Leftrightarrow\alpha_{ij}=0$$

is a partial order.

Conversely, suppose that a partial order $\leq$ is given on the set $\{1,2,\ldots,n\}$ and that the matrix $\mathcal{E}=\left(\alpha_{ij}\right)$ is defined by the formula above. Clearly, $\alpha_{ii}=0$ for all i. If $\alpha_{ik}=0$, then the ultra inequality

$$\alpha_{ik}\leq\max\left\{\alpha_{ij},\alpha_{jk}\right\}$$

holds automatically. Let $\alpha_{ik}=1$. This means that $i\not\leq k$. If both $\alpha_{ij}=0$ and $\alpha_{jk}=0$, then we would have $i\leq j$ and $j\leq k$, whence, by transitivity, $i\leq k$. This contradicts the condition $\alpha_{ik}=1$. Therefore at least one of the entries α_{ij} and α_{jk} is equal to 1, and hence

$$\max\left\{\alpha_{ij},\alpha_{jk}\right\}=1,$$

so the ultra inequality holds.

Reducedness also holds. If $i\neq j$, then by antisymmetry of the partial order the relations $i\leq j$ and $j\leq i$ cannot hold simultaneously. Therefore the two entries α_{ij} and α_{ji} cannot both be equal to zero. Hence

$$\alpha_{ij}+\alpha_{ji}\geq 1.$$

The corollary is proved.

Theorem 4.6. *For matrices* $A=\left(a_{ij}\right)$ *and* $B=\left(b_{ij}\right)$ *with nonnegative entries, define the minimax operation*

$$\left(A\odot B\right)_{ij}=\min_{1\leq k\leq n}\max\left\{a_{ik},b_{kj}\right\}.$$

Let $\mathcal{E}=\left(\alpha_{ij}\right)\in M_n\left(\mathbb{Z}_{\geq 0}\right)$ *be a matrix with zero main diagonal. Then* $\mathcal{E}$ *is an ultra exponent matrix if and only if*

$$\mathcal{E}\odot\mathcal{E}=\mathcal{E}.$$

Proof. Let $\mathcal{E}$ be an ultra matrix. Then for all i, j, k we have

$$\alpha_{ij} \le \max\left\{\alpha_{ik}, \alpha_{kj}\right\}.$$

Therefore,

$$\alpha_{ij} \le \min_{1\le k\le n} \max\left\{\alpha_{ik}, \alpha_{kj}\right\}.$$

On the other hand, if we take $k = i$, then

$$\max\left\{\alpha_{ii}, \alpha_{ij}\right\} = \max\left\{0, \alpha_{ij}\right\} = \alpha_{ij}.$$

Thus

$$\min_{1\le k\le n} \max\left\{\alpha_{ik}, \alpha_{kj}\right\} \le \alpha_{ij}.$$

Combining the two inequalities, we obtain

$$\min_{1\le k\le n} \max\left\{\alpha_{ik}, \alpha_{kj}\right\} = \alpha_{ij}.$$

Hence $\mathcal{E} \odot \mathcal{E} = \mathcal{E}$.

Conversely, suppose that $\mathcal{E} \odot \mathcal{E} = \mathcal{E}$. Then for all i, j we have

$$\alpha_{ij} = \min_{1\le k\le n} \max\left\{\alpha_{ik}, \alpha_{kj}\right\}.$$

It follows that, for every k,

$$\alpha_{ij} \le \max\left\{\alpha_{ik}, \alpha_{kj}\right\}.$$

This is exactly the ultra inequality. Therefore $\mathcal{E}$ is an ultra matrix. **The theorem is proved.**

Corollary 4.7. *Let A = (α_{ij}) be an ultra exponent matrix. For any distinct vertices i and j, the entry α_{ij} is equal to the corresponding minimax directed distance:*

$$\alpha_{ij} = \min_{P:i\to j} \max_{(u,v)\in P} \alpha_{uv},$$

where the minimum is taken over all directed paths from i to j in the complete directed graph on the vertex set, and the weight of an edge (u, v) is the matrix entry α_{uv}.

Proof. The inequality

$$\min_{P:i\to j} \max_{(u,v)\in P} \alpha_{uv} \le \alpha_{ij}$$

follows from the fact that among all paths there is a path of length 1 going directly from i to j.

We prove the opposite inequality. Let

$$P : i = i_0 \to i_1 \to \cdots \to i_m = j$$

be an arbitrary path. Applying the ultra inequality successively, we obtain

$$\alpha_{ij} \le \max\left\{\alpha_{i_0 i_1}, \alpha_{i_1 i_2}, \ldots, \alpha_{i_{m-1} i_m}\right\}.$$

Thus α_{ij} does not exceed the maximum weight of an edge on any path from i to j. Therefore α_{ij} does not exceed the minimum of such maxima over all paths. Together with the previous inequality, this gives the required equality. **The corollary is proved.**

Theorem 4.8. *Let* $\mathcal{E}=\left(\alpha_{ij}\right)$ *and* $\mathcal{F}=\left(\beta_{ij}\right)$ *be ultra exponent matrices of the same size. Define the matrix* $\mathcal{H}=\left(h_{ij}\right)$ *by*

$$h_{ij}=\max\left\{\alpha_{ij},\beta_{ij}\right\}.$$

Then $\mathcal{H}$ *is also an ultra exponent matrix.*

Proof. Since $\mathcal{E}$ and $\mathcal{F}$ are ultra matrices, for all i, j, k we have

$$\alpha_{ik}\le\max\left\{\alpha_{ij},\alpha_{jk}\right\},\quad \beta_{ik}\le\max\left\{\beta_{ij},\beta_{jk}\right\}.$$

Then

$$h_{ik}=\max\left\{\alpha_{ik},\beta_{ik}\right\}\le\max\left\{\alpha_{ij},\alpha_{jk},\beta_{ij},\beta_{jk}\right\}.$$

But

$$\max\left\{\alpha_{ij},\beta_{ij}\right\}=h_{ij},\ \max\left\{\alpha_{jk},\beta_{jk}\right\}=h_{jk}.$$

Therefore,

$$h_{ik}\le\max\left\{h_{ij},h_{jk}\right\}.$$

Moreover,

$$h_{ii}=\max\left\{\alpha_{ii},\beta_{ii}\right\}=0.$$

Let us also verify reducedness. For $i\ne j$ we have

$$h_{ij}+h_{ji}\ge\alpha_{ij}+\alpha_{ji}\ge 1,$$

because $\mathcal{E}$ is a reduced exponent matrix. Hence $\mathcal{H}$ is a reduced ultra exponent matrix. **The theorem is proved.**

Conclusions. The paper investigates ultra exponent matrices, that is, matrices for which the usual triangle inequality is strengthened to an ultrametric inequality. It is proved that every ultra exponent matrix is an exponent matrix, and that every exponent matrix with entries 0 and 1 is an ultra matrix.

It is shown that elementary transformations of the first type do not preserve the property of being an ultra matrix in general, whereas elementary transformations of the second type preserve it. It is proved that a quiver obtained from an ultra matrix having at least one entry greater than 1 is not rigid.

It is also established that not every admissible quiver with a loop at every vertex can be obtained from an ultra exponent matrix. This shows that the class of quivers corresponding to ultra exponent matrices is a proper subclass of the class of admissible quivers with loops at all vertices.

In addition, several structural characterizations of ultra matrices are obtained. In particular, it is proved that monotone deformations of values preserving 0 and 1 do not change the quiver of an ultra matrix. A filtration characterization of ultra matrices in terms of transitivity of the relations R_r is established; the partial order generated by zero entries of a reduced ultra matrix is described; and a classification of reduced $0,1$-ultra matrices in terms of partial orders is obtained. Furthermore, minimax idempotency of ultra matrices and the closure of their class under componentwise maximum are proved.

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УЛЬТРА МАТРИЦІ ПОКАЗНИКІВ

У статті досліджено ультра матриці показників, тобто зведені матриці показників, для яких звичайна трикутникова нерівність посилюється до нерівності ультраметричного типу. Доведено, що кожна ультра матриця показників є матрицею показників, а кожна матриця показників з елементами з множини $\{0, 1\}$ є ультра матрицею показників. Проаналізовано поведінку таких матриць відносно елементарних перетворень: перетворення першого типу загалом не зберігає ультравластивість, тоді як одночасна перестановка рядків і стовпців її зберігає. Показано, що сагайдак, одержаний зі зведеної ультра матриці показників, яка має хоча б один елемент, більший за одиницю, не є жорстким. Наведено контрприклад, який показує, що не кожний допустимий сагайдак із петлею в кожній вершині може бути одержаний з ультра матриці показників. Також встановлено низку структурних характеристик: описано монотонні деформації, які не змінюють відповідний сагайдак, фільтраційну характеристику через транзитивні порогові відношення, зв'язок 0-1 ультра матриць із частковими порядками, мінімаксну інтерпретацію та замкненість класу ультра матриць відносно покомпонентного максимуму. З погляду математичного моделювання ультра матриці показників можна розглядати як дискретні моделі напрямлених відстаней із обмеженнями вузького місця, де значення переходу визначається найсильнішим обмеженням на допустимих шляхах, а не сумарною вартістю. Це дає змогу застосовувати їх до моделювання ієрархічних систем, пріоритетних відношень, мінімаксної оптимізації, обмежених мережевих потоків, кластеризаційних структур та алгебраїчних або комбінаторних моделей, інформація яких кодується допустимими зваженими сагайдаками. Одержані результати дають інструменти для порівняння таких моделей і усунення надлишкових представлень без втрати напрямленої структури.

Ключові слова: *матриця показників, зведена матриця показників, допустимий сагайдак, ультра матриця показників, жорсткий сагайдак, вагова функція, математичне моделювання.*